

INTEGRAL EULER CHARACTERISTIC OF $\text{Out } F_{11}$

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ABSTRACT. We show that the integral Euler characteristic of the outer automorphism group of the free group of rank 11 is -1202 .

1. INTRODUCTION

This paper is a continuation of the previous paper [13], where we computed some parts of the Euler characteristics of three types of symplectic derivation Lie algebras.

Let $\mathfrak{h}_{g,1}$ be the symplectic derivation Lie algebra of the free Lie algebra $\mathcal{L}(H)$ generated by the fundamental representation H over \mathbb{Q} of the symplectic group $\text{Sp}(2g, \mathbb{Q})$. We may regard H as a representation of the corresponding Lie algebra $\mathfrak{sp}(2g, \mathbb{Q})$. Topologically, the vector space H is the first rational homology group of a compact connected oriented surface of genus g with one boundary component. The Lie algebra $\mathfrak{h}_{\infty,1}$ obtained from $\mathfrak{h}_{g,1}$ by taking the direct limit with respect to g is one of the three infinite dimensional Lie algebras considered by Kontsevich in [10, 11]. In these papers, he proved that, for $\mathfrak{h}_{\infty,1}$ named the *Lie case*, the homology group of $\mathfrak{h}_{\infty,1}$ is isomorphic to the free graded commutative algebra generated by the stable homology group of $\mathfrak{sp}(2g, \mathbb{Q})$ together with the totality of the cohomology groups of the outer automorphism groups $\text{Out } F_n$ of free groups F_n of rank $n \geq 2$. This is done by a deep consideration on the relationship between the cell structure of the outer space given by Culler and Vogtmann [4] and the chain complex which computes the graph homology associated with the Lie cyclic operad. The remaining two cases are named the *associative case* and the *commutative case*. Their homology groups are also related to other interesting geometrical objects such as cohomology groups of moduli spaces of Riemann surfaces and invariants of three dimensional manifolds etc.

Finding non-trivial rational (co)homology classes of $\text{Out } F_n$ has been a difficult problem. A striking result by Galatius [5] shows that there are no rational *stable* reduced (co)homology classes. At present, only the three *unstable* classes, which are the first three of a series of classes introduced by the first author in [12], are shown to be non-trivial (see Conant and Vogtmann [3] and Gray [7]). Here we would like to mention that in a recent paper by Conant, Kassabov and Vogtmann [1], they constructed new homology classes of $\text{Out } F_n$ which are related to the theory of elliptic modular forms, although their non-triviality is unknown. The situation being like this, it had been an important problem to prove the existence of non-trivial classes. In this context, our work in [13] of determining some parts of the integral Euler characteristics showed

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that there exist at least one hundred new non-trivial classes and it also suggested that further computations should reveal the existence of many more classes.

More specifically, the Lie algebra $\mathfrak{h}_{g,1}$ has a grading induced from that of the free Lie algebra $\mathcal{L}(H) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(H)$, so that we have a direct sum decomposition $\mathfrak{h}_{g,1} = \bigoplus_{k=0}^{\infty} \mathfrak{h}_{g,1}(k)$. Here $\mathfrak{h}_{g,1}(k)$ is the degree k homogeneous part and $[\mathfrak{h}_{g,1}(k_1), \mathfrak{h}_{g,1}(k_2)] \subset \mathfrak{h}_{g,1}(k_1 + k_2)$ holds for any k_1 and k_2 . The explicit description of $\mathfrak{h}_{g,1}(k)$ is given by

$$\mathfrak{h}_{g,1}(k) = \text{Ker} \left(H \otimes \mathcal{L}_{k+1}(H) \xrightarrow{[\cdot, \cdot]} \mathcal{L}_{k+2}(H) \right).$$

Since the bracket operation $[\cdot, \cdot]$ on $\mathcal{L}(H)$ is equivariant with respect to the natural action of $\text{Sp}(2g, \mathbb{Q})$, the space $\mathfrak{h}_{g,1}(k)$ becomes an $\text{Sp}(2g, \mathbb{Q})$ -module. It is known that the homology of the graded Lie algebra $\mathfrak{h}_{g,1}$ has another grading. That is, we have a direct sum decomposition

$$H_*(\mathfrak{h}_{g,1}) = \bigoplus_{w=0}^{\infty} H_*(\mathfrak{h}_{g,1})_w$$

with $H_*(\mathfrak{h}_{g,1})_w$ obtained as the homology of the subcomplex generated by the chains in $\wedge^* \mathfrak{h}_{g,1}$ of total degree w and called the *weight w part* hereafter (see [13, Section 2]).

Our main concern is the computation of $H_*(\mathfrak{h}_{\infty,1})_w$ after taking the direct limit with respect to g . At first glance, it may look too huge to handle. However, the following observation shows that it is not necessarily so. Let $\mathfrak{h}_{g,1}^+ = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$ be the ideal of the *positive* degree part. The spaces $H_*(\mathfrak{h}_{g,1})_w$ and $H_*(\mathfrak{h}_{g,1}^+)_w$ are also $\text{Sp}(2g, \mathbb{Q})$ -modules. As stated by Kontsevich [10] and proved in detail by Conant and Vogtmann [2, Proposition 8] (see also [13, Section 2]), we have

$$H_*(\mathfrak{h}_{g,1})_w = H_*(\mathfrak{h}_{g,1}^+)_w^{\text{Sp}}$$

for any $w \geq 1$. Here, for an $\text{Sp}(2g, \mathbb{Q})$ -module V , we denote by V^{Sp} the invariant part for the $\text{Sp}(2g, \mathbb{Q})$ -action. The general theory of $\text{Sp}(2g, \mathbb{Q})$ -representations says that the invariant part $H_*(\mathfrak{h}_{g,1}^+)_w^{\text{Sp}}$ as well as that of the corresponding chain complex stabilizes when g becomes large. In particular, they are *finite* dimensional. Then Kontsevich's theorem says that the isomorphism

$$PH_k(\mathfrak{h}_{\infty,1})_{2n} \cong H^{2n-k}(\text{Out } F_{n+1}; \mathbb{Q})$$

holds for $n \geq 1$ and $k \geq 1$, where $PH_k(\mathfrak{h}_{\infty,1})_{2n}$ is the primitive part in $H_k(\mathfrak{h}_{\infty,1})_{2n}$ with respect to the commutative and co-commutative Hopf algebra structure (see [2, Section 2]).

In our previous paper [13], we determined the dimensions of the chain complex $C_i(\mathfrak{h}_{\infty,1}^+)^{\text{Sp}}$ which computes the Sp -invariant homology of the Lie algebra $\mathfrak{h}_{\infty,1}^+$ up to weight 18. From this, we obtained the value of the Euler characteristic

$$\chi(H_*(\mathfrak{h}_{\infty,1}^+)_w^{\text{Sp}}) = \sum_{i=1}^w (-1)^i \dim(C_i(\mathfrak{h}_{\infty,1}^+)_w^{\text{Sp}})$$

of each weight $w \leq 18$ summand. The result is given as follows:

Theorem 1.1 ([13]). *The Euler characteristics $\chi(H_*(\mathfrak{h}_{\infty,1}^+)_w^{\text{Sp}})$ of the Sp -invariant homology groups of the Lie algebra $\mathfrak{h}_{\infty,1}^+$ up to weight $w \leq 18$ are given by the following table:*

w	2	4	6	8	10	12	14	16	18
$\chi(H_*(\mathfrak{h}_{\infty,1}^+)^{\text{Sp}}_w)$	1	2	4	6	10	16	23	13	-96

Note that $C_i(\mathfrak{h}_{\infty,1}^+)^{\text{Sp}}_w$ is trivial if w is odd. By combining Theorem 1.1 with the description of the generators of the stable cohomologies due to Kontsevich, we obtain the following result:

Theorem 1.2 ([13]). *The integral Euler characteristics*

$$e(\text{Out } F_n) = \sum_{i=0}^{2n-3} (-1)^i \dim(H^i(\text{Out } F_n; \mathbb{Q}))$$

of $\text{Out } F_n$ up to $n \leq 10$ are given as follows:

n	2	3	4	5	6	7	8	9	10
$e(\text{Out } F_n)$	1	1	2	1	2	1	1	-21	-124

By Theorem 1.2, the existence of non-trivial *odd* dimensional rational cohomology classes of $\text{Out } F_n$ was shown for the first time.

The purpose of the present paper is to extend our results to the next step, weight 20, by which we determine the integral Euler characteristic of $\text{Out } F_{11}$. The details of our explicit computation will be given in Section 2. We will also compare our results with the result of Smillie and Vogtmann [15] on the rational (or orbifold) Euler characteristics of $\text{Out } F_n$.

For the computations of this paper, we basically used the same methods as in [13, Section 4], namely we made intensive use of well known software LiE and Mathematica. Since our computations heavily depend on computers, the checking process for the accuracy is as important as the actual computational process. Our checking methods were also discussed in [13, Section 4].

Finally, we comment about related works on the other two cases of Kontsevich's theorem. As for the commutative case, Willwacher and Živković [17] recently obtained the generating function of the (total) Euler characteristic and computed the explicit values up to weight 60. Our former results in [13] are consistent with theirs. For the associative case, we can apply Gorsky's formula [6] for the equivariant Euler characteristics of moduli spaces of Riemann surfaces with marked points. The formula enables the authors to compute the Euler characteristics up to weight 250 [14], which coincide with our former computation up to weight 16 in [13]. These facts would support the accuracy of our computations in this paper since many parts of the data on various symplectic modules we used are in common with those for the commutative and associative cases. On the other hand, no result is known about the generating function for the Lie case which is similar to Gorsky's formula for the associative case. There seem to exist difficulties peculiar to this case, which add an additional meaning to our computational results.

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2. MAIN RESULTS

We compute the dimension of the chain complex $C_i = C_i(\mathfrak{h}_{\infty,1}^+)_{20}^{\text{Sp}}$ for $H_i(\mathfrak{h}_{\infty,1}^+)_{20}^{\text{Sp}}$ explicitly. More precisely, we determine the dimension of the finite dimensional complex:

$$C_i = \bigoplus_{\substack{i_1 + \dots + i_{20} = i \\ i_1 + 2i_2 + \dots + 20i_{20} = 20}} (\wedge^{i_1}(\mathfrak{h}_{\infty,1}^+(1)) \otimes \wedge^{i_2}(\mathfrak{h}_{\infty,1}^+(2)) \otimes \dots \otimes \wedge^{i_{20}}(\mathfrak{h}_{\infty,1}^+(20)))^{\text{Sp}}$$

where $\mathfrak{h}_{\infty,1}^+(k)$ is the degree k part of $\mathfrak{h}_{\infty,1}^+$. The result is shown in Table 1.

TABLE 1. The dimension of C_i

	dimension
C_1	29729988
C_2	410769138
C_3	2864009351
C_4	13262053269
C_5	45353489325
C_6	120900142805
C_7	259222260499
C_8	455821729958
C_9	665350325867
C_{10}	811759271904
C_{11}	830129318093
C_{12}	711071098888
C_{13}	508080341074
C_{14}	300343387403
C_{15}	144874973588
C_{16}	55809757570
C_{17}	16607403485
C_{18}	3615255878
C_{19}	519201462
C_{20}	37584620
total	4946062104165
χ	-1299

Proposition 2.1 ([13, Proposition 6.2]). *The weight generating function, denoted by $h(t)$, for the Sp-invariant stable homology group $H_*(\mathfrak{h}_{\infty,1}^+)_{\text{Sp}}$ is given by*

$$h(t) = \prod_{n=2}^{\infty} (1 - t^{2n-2})^{-e(\text{Out } F_n)}$$

where $e(\text{Out } F_n)$ denotes the integral Euler characteristic of $\text{Out } F_n$.

Theorem 2.2.

$$e(\text{Out } F_{11}) = -1202.$$

Proof. By Theorem 1.1 and Table 1, the weight generating function

$$h(t) = \sum_{w=0}^{\infty} \chi(H_*(\mathfrak{h}_{\infty,1}^+)_w^{\text{Sp}}) t^w$$

is written as

$$h(t) = 1 + t^2 + 2t^4 + 4t^6 + 6t^8 + 10t^{10} + 16t^{12} + 23t^{14} + 13t^{16} - 96t^{18} - 1299t^{20} + \dots$$

By using the same method in [13], we can determine the Euler characteristics of the primitive parts, namely $e(\text{Out } F_{11})$. To be precise, the Euler characteristic of lower terms in the weight 20 is -97 . Then the Euler characteristics of the primitive parts is $-1299 - (-97) = -1202$. In other words, if we consider

$$\begin{aligned} \bar{h}(t) = & (1 - t^2)^{-1}(1 - t^4)^{-1}(1 - t^6)^{-2}(1 - t^8)^{-1}(1 - t^{10})^{-2} \\ & (1 - t^{12})^{-1}(1 - t^{14})^{-1}(1 - t^{16})^{21}(1 - t^{18})^{124}(1 - t^{20})^{1202}, \end{aligned}$$

then $\bar{h}(t)$ is congruent to $h(t)$ modulo t^{21} . By Proposition 2.1, we conclude that

$$e(\text{Out } F_{11}) = -1202.$$

□

The fourth row of Table 2 is the Euler characteristic of the primitive part which gives us the Euler characteristic of Out F_n .

TABLE 2. Numbers of new generators for $H_*(\mathfrak{h}_{\infty,1}^+)_w^{\text{Sp}}$

w	2	4	6	8	10	12	14	16	18	20
χ	1	2	4	6	10	16	23	13	-96	-1299
χ of lower terms	0	1	2	5	8	15	22	34	28	-97
χ of primitive part	1	1	2	1	2	1	1	-21	-124	-1202

As a remarkable corollary, we see that there exist at least 1203 *odd* dimensional non-trivial rational cohomology classes of Out F_{11} .

By using this result, we can extend our former table in [13] to Table 3, which compares the *rational* and the *integral* Euler characteristics of Out F_n . The second row of Table 3 is the rational Euler characteristics $\chi(\text{Out } F_n)$ of Out F_n given by Smillie and Vogtmann [15], written up to the second decimal places here. The third row is the integral Euler characteristics $e(\text{Out } F_n)$ of Out F_n given by Theorems 1.2 and 2.2.

Here we would like to mention the following two important open problems, which show a considerable difference between the Lie case and the other two cases. One is the asymptotic behavior of the rational Euler characteristics of Out F_n . Smillie and Vogtmann [15, Section 6] conjectured that the rational Euler characteristics of Out F_n are negative for all n , which holds for $n \leq 100$ as mentioned in Vogtmann [16], and their absolute values grow exponentially with n . However, this conjecture is not settled yet. The other is the problem of determining whether the ratio of the rational Euler

characteristics and the integral one tends to 1 or not. In the case of the moduli spaces of Riemann surfaces, Harer and Zagier [8] proved that the ratio tends to 1 asymptotically.

TABLE 3. χ versus e for $\text{Out } F_n$

n	2	3	4	5	6	7	8	9	10	11
χ	−0.04	−0.02	−0.02	−0.06	−0.20	−0.87	−4.58	−28.52	−205.83	−1690.70
e	1	1	2	1	2	1	1	−21	−124	−1202

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